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ON THE BUCKLING AND IMPERFECTION-SENSITIVITY OF ARCHES WITH AND WITHOUT PRESTRESS

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Abstract—Shallow, essentially inextensional, simply-supported arches under central point load are used extensively to illustrate the cusped imperfection-sensitivity of an unstable-symmetric bifurcation. Often these are made in the laboratory by simply buckling a straight strut between fixed abutments, and are therefore prestressed. We give here a concise buckling and post-buckling analysis of such an arch with any degree of prestress, and show that the critical bifurcation load varies linearly with the prestress. The immediate use of the inextensibility greatly simplifies the analysis and leads to neat closed-form solutions which agree extremely well with the limited theoretical and experimental results available from other sources.

1. INTRODUCTION

Following the classical work of Roorda [1–4] at University College London, very thin simply-supported steel arches under central point load are used extensively to illustrate the cusped imperfection-sensitivity of an unstable-symmetric point of bifurcation[5, 6]. These are sometimes rolled, to give a pure stress-free arch, but are more often made simply by buckling a straight strut to give a pre-stressed arch.

Roorda tested an arch of each type, but only gave an analysis of the pure stress-free arch. His theory for this employed two fourier harmonics for the deflected form, and included the axial stiffness EA, the product of Young's Modulus and the cross-sectional area. This axial compressibility gives a non-trivial fundamental path before buckling which greatly complicates the analysis, while it is clear from the dimensions of his test arch that this compressibility is exceedingly small and could validly be neglected. Indeed the axial compressibility can be equated to zero in his final results by asymptotically setting EA to infinity: this greatly simplifies the form of solution, without noticeably altering the numerical values for the type of arch under discussion.

In this paper we give a theory for a shallow arch with any degree of pre-stress (including zero) which uses axial inextensibility from the start. The arch has then a purely trivial fundamental path with no deflection before buckling (under a two-harmonic approximation) allowing a neat closed form analysis for the critical load P^C and the equations of imperfection-sensitivity.

For the pure, stress-free arch, a comparison can be made with the solution of Roorda, which can be found unchanged in the book of Huseyin[6]. Our solution for P^C,

$$P^C = 2\pi^4 \frac{EIH}{L^3}$$

agrees in form with Roorda's asymptotic result which has the alternative numerical factor 64π: Roorda's P^C is therefore 1.032 times our own. Here EI is the bending stiffness of the arch, H is its rise and L is its curved length. When normalized with respect to the corresponding P^C, our equation of imperfection-sensitivity is

$$\frac{P}{P^C} = 1 - 3^{4/3}2^{-5/3}\pi^{2/3}\epsilon^{2/3}$$

which is, remarkably, *identical* to that of Roorda. Here P is the load-carrying capacity of an imperfect arch in which the load is off-set from the centre-line a distance ϵL.

For the pre-stressed arch made from an initially straight strut, no theory seems to be available for comparison, but our

$$P^C = \frac{3}{2}\pi^4 \frac{EIH}{L^3}$$

agrees well with experimental results, as does our imperfection-sensitivity

$$\frac{P}{P^C} = 1 - 3 \times 2^{-1} \pi^{2/3} \epsilon^{2/3}.$$

The present succinct analysis thus gives us results agreeing well with existing theory and experiment. It is simpler than that of Roorda and Huseyin by virtue of its trivial fundamental solution. It might of course be argued that the experimental arches do indeed deflect symmetrically before buckling, but this is almost entirely due to third and higher harmonics rather than the minute axial compressibility. These higher harmonics would have to be taken into account in a more precise solution.

Our analysis shows for the first time the effect of initial pre-stress, and it is interesting to note that P^C decreases linearly with pre-stress as measured by the initial downwards stress-free amplitude A_0. This continues until an arch with $A_0 = 3H$ buckles at zero applied load (within the present approximation), and an arch with greater A_0 would jump spontaneously from its unstable unloaded state.

2. INEXTENSIONAL THEORY

Our aim here is to analyse shallow inextensional arches, pinned to fixed abutments and loaded by a dead vertical load nominally at the centre. However, since we wish to embrace within a single formulation, pure arches, buckled struts and even arches which would be stress-free in an inverted configuration, it seems convenient to employ our earlier strut formulation[5]. The abutments then simply impose a constant value of the end-shortening function.

To deal with the prestress, the strut is presumed to be deformed into an initial bent state, and is then imagined to be stress-relieved: it is subsequent changes of curvature from this initial state that induce strain energy and associated bending moments.

After a Fourier series expansion of the normal deflection, only the leading two terms are retained. The inextensibility of the centre-line together with the constraint imposed by the fixed supports then reduce the degree of freedom to one. In fact the state of the arch is constrained to lie on a closed curve in the space of the two harmonic amplitudes, akin to the neat topological study of strut and arch buckling due to Zeeman[7] which inspired this present work.

2.1 *Strain energy*

For the strain energy we proceed in the same way as for the inextensional strut of ([5] p 28), but allowing for the extra effect of full stress-relief in a general deformed state. Let us consider the initially straight, simply-supported, strut of length L shown in Fig. 1, and oblige it to undergo an end displacement $\mathscr{E}$ in the axial direction as shown. We assume that the strut is inextensional, and has a flexural stiffness EI.

The curvature of a small element is given by

$$\chi = \frac{d\theta}{dx} = \frac{d}{dx}\sin^{-1} w' = w''(1 - w'^2)^{-1/2},$$

where a prime denotes differentiation with respect to x. We have only strain energy of bending to consider, which is given by

$$dU = \frac{1}{2} M\, d\theta = \frac{1}{2} EI\chi\, d\theta = \frac{1}{2} EI\chi^2\, dx.$$

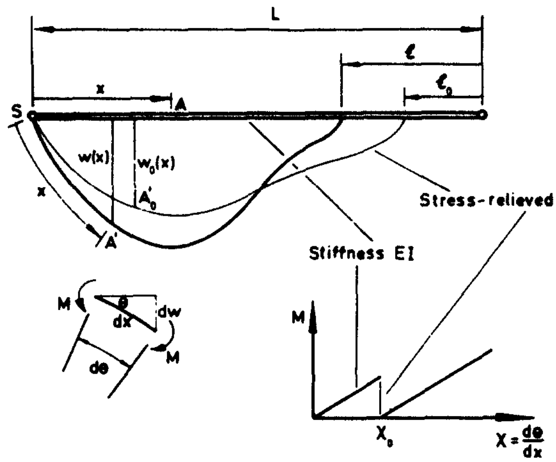

Fig. 1. General deformed shape of the inextensional arch, and the stress-relieved state.

We now suppose that, at a certain deformed shape $w_0(x)$ with end displacement $\mathscr{E}_0$ as shown, the system is *stress-relieved*, such that the elemental bending moment M and the stored strain energy drop to zero. The moment-curvature relationship takes the form shown in Fig. 1 and we can see that the elemental strain energy for a general deformed shape $w(x)$ is now,

$$dU = \frac{1}{2} EI(\chi - \chi_0)^2 \, dx,$$

where χ_0 is the curvature in the stress-relieved state. Integrating, we can now write the total strain energy as,

$$\begin{aligned} U &= \frac{1}{2} EI \int_0^L (\chi - \chi_0)^2 \, dx \\ &= \frac{1}{2} EI \int_0^L [w''^2(1 - w'^2)^{-1} - 2w'' w_0''(1 - w'^2)^{-1/2}(1 - w_0'^2)^{-1/2} \\ &\quad + w_0''^2(1 - w_0'^2)^{-1}] \, dx. \end{aligned}$$

Expanding all these terms as power series, we have,

$$\begin{aligned} U = \frac{1}{2} EI \int_0^L \Big[&(w_0''^2 + w_0''^2 w_0'^2 + w_0''^2 w_0'^4 + \cdots) \\ &-2w_0''\left(1 + \frac{1}{2} w_0'^2 + \frac{3}{8} w_0'^4 + \cdots\right) w'' + w''^2 \\ &-w_0''\left(1 + \frac{1}{2} w_0'^2 + \frac{3}{8} w_0'^4 + \cdots\right) w'' w'^2 + w''^2 w'^2 + \cdots \Big] \, dx, \end{aligned} \tag{1}$$

which is arranged in ascending powers of w and its derivatives. For a linear eigenvalue study we would need merely the quadratic terms.

The corresponding total end constraint $\mathscr{E}$ is given by,

$$\begin{aligned} \mathscr{E} &= L - \int_0^L \cos\theta \, dx \\ &= \int_0^L \left(\frac{1}{2} w'^2 + \frac{1}{8} w'^4 + \frac{1}{16} w'^6 + \cdots\right) dx, \end{aligned} \tag{2}$$

as in our book. Again, a linear eigenvalue study needs just the leading term.

2.2 *Expansion in Fourier harmonics*

After the general formulation we now restrict the deflection functions $w_0(x)$ and $w(x)$ to appropriate Fourier harmonics. We suppose that the arch is initially deflected upwards with a rise H at the crown, and is loaded perfectly centrally by a dead vertical load P. Later we test for the effects of imperfections by allowing a small offset of the load, but for the moment we restrict the study to this *perfect system.*

We start by assuming that in the stress-relieved state, before the arch is fixed between the rigid supports, the deflection $w_0(x)$ is a half-sine wave as shown at the top of Fig. 2. We thus write,

$$w_0 = A_0 \sin \frac{\pi x}{L},$$

where A_0, like w_0, is positive measured downwards. This allows for the study of a variety of states of initial pre-stress, by altering the value of A_0. We note in passing that a symmetry-destroying imperfection could be introduced here, in the form of pre-stress, with the addition of a small second harmonic term.

For a full harmonic analysis of the arch we could expand w as

$$w(x) = \sum_{i=1}^{\infty} Q_i \sin \frac{i\pi x}{L}.$$

However, we truncate the series after the leading two terms, and write

$$w(x) = Q_1 \sin \frac{\pi x}{L} + Q_2 \sin \frac{2\pi x}{L}, \tag{3}$$

where Q_1 and Q_2, like w, are positive measured downwards. This reduces the system to one which at first seems to have two degrees of freedom. But the inextensibility and the constraint of the fixed supports impose a further condition so that Q_1 and Q_2 are not independent; we thus have some locus in Q_i space, $Q_1 = f(Q_2)$ say, which the system is obliged to follow, and it is reduced to a single degree of freedom.

This may seem a somewhat crude approximation to a continuum with an infinite number of degrees of freedom, but it is remarkably successful, as indicated first by the simple closed-form solutions obtained and secondly by the good agreement with experiments. It clearly closely

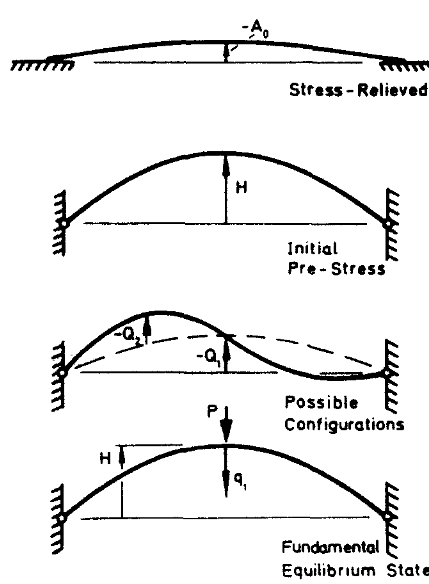

Fig. 2. Arch geometry and description.

resembles a single degree of freedom Rayleigh-Ritz approach. Its great virtue is its simplicity, especially in that it exhibits a trivial fundamental path of equilibria, $Q_2 = 0$, $Q_1 = -H = \text{const}$, for all loads P. This is a significant aid to solution, since the description of the nonlinear path for the true arch is in itself a considerable problem, and one which is largely irrelevant as far as the bifurcational response is concerned. These are important points, and we shall be returning to them at intervals later.

So, substituting the assumed forms for w_0 and w into the general strain energy function (1) we are left with a number of integrals of sinusoidal products to evaluate. These exhibit both quadratic and higher-order orthogonalities, and lead to the diagonalized strain energy function,

$$U(Q_1, Q_2) = \frac{1}{2}EI\left\{-\frac{\pi^4}{L^2}\left[\frac{A_0}{L} + \frac{\pi^2}{8}\left(\frac{A_0}{L}\right)^3 + \text{higher-order}\right]Q_1 + \frac{1}{2}\frac{\pi^4}{L^3}(Q_1^2 + 16Q_2^2) - \frac{1}{8}\frac{\pi^6}{L^4}\left(\frac{A_0}{L} + \text{higher-order}\right)(Q_1^3 + 8Q_1Q_2^2) + \frac{1}{8}\frac{\pi^6}{L^5}(Q_1^4 + 40Q_1^2Q_2^2 + 64Q_2^4) + \text{higher-order}\right\}, \tag{4}$$

which is arranged in ascending powers of Q_1 and Q_2. Here constant terms arising from just w_0 and its derivatives have been ignored since we are concerned with variations of the strain energy, never its absolute value.

2.3 *The constraint condition*

We now turn our attention to the constraint condition imposed by the rigid supports, and seek a way of expressing the resulting locus in $Q_1 - Q_2$ space. This is to be written as a Taylor series, first in the two variables, and secondly via an intrinsic perturbation scheme, in Q_2 alone. The latter is only appropriate if the locus is single-valued in Q_2, although during buckling Q_2 reaches a maximum and then falls again to zero as the arch finds its final, upsidedown, equilibrium configuration. The analysis is thus essentially localized, which falls into line with the philosophy of structural mechanics, where interest focusses more on the failure itself, than on the gross deformations of a failed system. The treatment can be seen as a useful quantitative complement to Zeeman's topological study[7].

The intrinsic perturbation scheme is a concept we have used at length in the general branching analyses of our book, and this simple example serves as an introduction to the more advanced studies. Only two variables are involved, and we can write all derivatives explicitly, without having to resort to the neater, but more obscure, subscript notation used elsewhere.

The constraint condition is simply $\mathscr{E} = \text{const.}$, where $\mathscr{E}$ is the end-shortening function (2). Substituting the assumed form for w, given by eqn (3), into this expression and performing the necessary integrations, we have,

$$\mathscr{E} = \frac{\pi^2}{4L}(Q_1^2 + 4Q_2^2) + \frac{3}{64}\frac{\pi^4}{L^3}(Q_1^4 + 16Q_1^2Q_2^2 + 16Q_2^4) + \text{higher-order terms.} \tag{5}$$

We see that $\mathscr{E}$ is diagonalized (no Q_1Q_2 cross term), and higher-order orthogonalities have set some of the quartics to zero.

We are seeking a Taylor expansion, and thus must start with coordinates measured from the unbuckled configuration. Introducing the incremental coordinate q_1 defined by

$$Q_1 = -H + q_1, \tag{6}$$

(see Fig. 2), we substitute this into eqn (5) to give,

$$\mathscr{E} = \frac{1}{4}\frac{\pi^2}{L}(q_1^2 - 2Hq_1 + 4Q_2^2) + \frac{3}{64}\frac{\pi^4}{L^3}(q_1^4 - 4Hq_1^3 + 6H^2q_1^2 - 4H^3q_1 + 16q_1^2Q_2^2 - 32Hq_1Q_2^2 + 16H^2Q_2^2 + 16Q_2^4) + \text{higher-order terms.} \tag{7}$$

Here as before, constant terms are ignored, since we are to be concerned just with variations of $\mathscr{C}$, never its absolute value. We see that the expression has become more complicated, with the appearance of linear and cubic terms.

But we have now a Taylor series, and can immediately write down the derivatives of $\mathscr{C}$, evaluated in the pre-buckled state $F(q_1 = Q_2 = 0)$, that we need later. We have for example,

$$\left.\frac{\partial \mathscr{C}}{\partial q_1}\right|^F = -\frac{1}{2}\frac{\pi^2 H}{L}\left[1+\text{order}\left(\frac{H}{L}\right)^2\right], \left.\frac{\partial \mathscr{C}}{\partial Q_2}\right|^F = 0,$$

$$\left.\frac{\partial^2 \mathscr{C}}{\partial q_1^2}\right|^F = \frac{1}{2}\frac{\pi^2}{L}\left[1+\text{order}\left(\frac{H}{L}\right)^2\right], \left.\frac{\partial^2 \mathscr{C}}{\partial Q_2^2}\right|^F = 2\frac{\pi^2}{L}\left[1+\text{order}\left(\frac{H}{L}\right)^2\right],$$

etc. and we need the cubic,

$$\left.\frac{\partial^3 \mathscr{C}}{\partial q_1 \partial Q_2^2}\right|^F = -3\frac{\pi^4 H}{L^3}\left[1+\text{order}\left(\frac{H}{L}\right)^2\right],$$

and the quartic,

$$\left.\frac{\partial^4 \mathscr{C}}{\partial Q_2^4}\right|^F = 18\frac{\pi^4}{L^3}\left[1+\text{order}\left(\frac{H}{L}\right)^2\right].$$

In all the following analysis only the leading term of each series need be considered, since we assume the arch to be of moderate rise, and hence,

$$\left(\frac{H}{L}\right)^2 \ll 1.$$

A simple perturbation scheme. We start the perturbation analysis by assuming that the system is constrained to follow a locus in coordinate space described by the parametric form,

$$q_1 = q_1(Q_2).$$

This is substituted into the constraint condition $\mathscr{C} = \text{const.} = K$, which then becomes an identity, since it is satisfied by all values of the single independent variable Q_2 in the region of interest. Thus,

$$\mathscr{C}[q_1(Q_2), Q_2] \equiv K. \tag{8}$$

This can be repeatedly differentiated with respect to Q_2. Evaluation of the resulting equations in the unbuckled state F then gives a series of sequentially linear problems, which are successively solved for derivatives of q_1 with respect to Q_2. These could be used to construct a Taylor series form of the constraint condition in the one variable Q_2, or we can use the derivatives directly as in the following post-buckling study.

We note that the process can only be applied to an identity, or we would be evaluating (on the locus of interest) before differentiating, thereby denying the system its full range of allowable deflection configurations.

Thus differentiating (8) repeatedly, we obtain,

$$\frac{d\mathscr{C}}{dQ_2} = \frac{\partial \mathscr{C}}{\partial q_1}\frac{dq_1}{dQ_2} + \frac{\partial \mathscr{C}}{\partial Q_2} = 0,$$

$$\frac{d^2\mathscr{C}}{dQ_2^2} = \frac{\partial^2 \mathscr{C}}{\partial q_1^2}\left(\frac{dq_1}{dQ_2}\right)^2 + 2\frac{\partial^2 \mathscr{C}}{\partial q_1 \partial Q_2}\frac{dq_1}{dQ_2} + \frac{\partial^2 \mathscr{C}}{\partial Q_2^2} + \frac{\partial \mathscr{C}}{\partial q_1}\frac{d^2 q_1}{dQ_2^2} = 0,$$

etc. In the following work we require the first four equations, but the last two are lengthy and are not given explicitly here.

Evaluating at F and using the known partial derivatives of $\mathscr{C}$, the first three equations give directly,

$$\left.\frac{dq_1}{dQ_2}\right|^F = 0, \quad \left.\frac{d^2q_1}{dQ_2^2}\right|^F = \frac{4}{H}, \quad \left.\frac{d^3q_1}{dQ_2^3}\right|^F = 0, \tag{9}$$

and the fourth equation is reduced to,

$$\left.\frac{\partial^4 \mathscr{C}}{\partial Q_2^4} + 6\frac{\partial^3 \mathscr{C}}{\partial q_1 \partial Q_2^2}\frac{d^2q_1}{dQ_2^2} + 3\frac{\partial^2 \mathscr{C}}{\partial q_1^2}\left(\frac{d^2q_1}{dQ_2^2}\right)^2 + \frac{\partial \mathscr{C}}{\partial q_1}\frac{d^4q_1}{dQ_2^4}\right|^F = 0. \tag{10}$$

Substituting the known derivatives, the first two terms can be neglected according to the moderate rise approximation, $(H/L)^2 \ll 1$. This leads to the simple result for the fourth derivative,

$$\left.\frac{d^4q_1}{dQ_2^4}\right|^F = \frac{48}{H^3}. \tag{11}$$

It is interesting to observe that this could have been achieved using no higher than quadratic derivatives of $\mathscr{C}$. Here we abandon the scheme, although further derivatives can be found by the obvious extension, if necessary.

2.4 *Linear eigenvalue analysis*

Before we proceed with the eigenvalue analysis to determine the critical load of the arch, we first write down the full potential energy expression, including the nonlinear terms required later in the post-buckling analysis. Combining the strain energy (4) and the work done by the load $(-Pq_1)$, and adopting the earlier incremental transformation (6) so we have a Taylor series as before, the potential energy function can be written,

$$\begin{aligned} V = \frac{1}{2}EI\Bigg\{&-\frac{\pi^4}{L^2}\left[\frac{A_0}{L}+\frac{H}{L}+\frac{\pi^2}{8}\left(\frac{A_0}{L}\right)^3+\frac{3}{8}\pi^2\frac{A_0}{L}\left(\frac{H}{L}\right)^2+\frac{\pi^2}{2}\left(\frac{H}{L}\right)^3\right]q_1 \\ &+\frac{1}{2}\frac{\pi^4}{L^3}(q_1^2+16Q_2^2)-\frac{1}{8}\frac{\pi^6}{L^4}\left(\frac{A_0}{L}+4\frac{H}{L}\right)q_1^3-\frac{\pi^6}{L^4}\left(\frac{A_0}{L}+10\frac{H}{L}\right)q_1Q_2^2 \\ &+\frac{1}{8}\frac{\pi^6}{L^5}(q_1^4+40q_1^2Q_2^2+64Q_2^4)+\text{higher-order terms}\Bigg\}-Pq_1. \end{aligned} \tag{12}$$

Here as before constant terms have been ignored, and we have assumed that the arch in both the stress-relieved and pre-stressed states is of moderate rise, thereby neglecting $(A_0/L)^2$, $(H/L)^2$, and A_0H/L^2 in comparison with unity. However, higher-order terms have been included in the coefficient of the linear q_1 term, since we shall later investigate specifically an arch with no initial pre-stress, for which the leading term vanishes; this turns out to be an unnecessary precaution, but for the moment we retain the terms.

For any constant load P this is a Taylor series in q_1 and Q_2, and we can immediately write down all the partial derivatives that we shall need later. We have first,

$$\begin{gathered} \left.\frac{\partial V}{\partial q_1}\right|^F = -\frac{1}{2}EI\frac{\pi^4}{L^2}\left\{\frac{A_0}{L}+\frac{H}{L}+\frac{\pi^2}{8}\left(\frac{A_0}{L}\right)^3+\frac{3}{8}\pi^2\frac{A_0}{L}\left(\frac{H}{L}\right)^2+\frac{\pi^2}{2}\left(\frac{H}{L}\right)^3\right\}-P, \\ \left.\frac{\partial V}{\partial Q_2}\right|^F = 0, \quad \left.\frac{\partial^2 V}{\partial q_1^2}\right|^F = \frac{1}{2}EI\frac{\pi^4}{L^3}, \quad \left.\frac{\partial^2 V}{\partial q_1 \partial Q_2}\right|^F = 0, \quad \left.\frac{\partial^2 V}{\partial Q_2^2}\right|^F = 8\,EI\frac{\pi^4}{L^3}, \end{gathered} \tag{13}$$

etc. We shall also need the cubic coefficient,

$$\left.\frac{\partial^3 V}{\partial q_1 \partial Q_2^2}\right|^F = -EI\frac{\pi^6}{L^4}\left(\frac{A_0}{L}+10\frac{H}{L}\right), \tag{14}$$

and the quartic,

$$\frac{\partial^4 V}{\partial Q_2^4}\bigg|^F = 96\, EI\frac{\pi^6}{L^5}, \tag{15}$$

in the post-buckling analysis.

It must be remembered that configurations of the arch are subject to the constraint condition, so q_1 and Q_2 as they appear here are not valid generalized coordinates, the system having just one degree of freedom. However, remembering the earlier parametric representation of the constraint condition, $q_1 = q_1(Q_2)$, we can substitute this and regard Q_2 as the single valid variable; we note that the function is already known implicitly, as derivatives evaluated at the unbuckled state F.

Thus writing V as,

$$V = V[q_1(Q_2), Q_2],$$

at any fixed load level, we differentiate with respect to Q_2 to obtain the equilibrium equation,

$$\frac{\mathrm{d}V}{\mathrm{d}Q_2} = \frac{\partial V}{\partial q_1}\frac{\mathrm{d}q_1}{\mathrm{d}Q_2} + \frac{\partial V}{\partial Q_2} = 0. \tag{16}$$

If we evaluate this equation at F we find on substitution from eqns (9) and (13) that the condition is identically satisfied for all P. This confirms that there is a trivial fundamental equilibrium state $F(Q_1 = -H, Q_2 = 0)$, for any load value.

To check the stability of this equilibrium state we form the second derivative,

$$\frac{\mathrm{d}^2 V}{\mathrm{d}Q_2^2} = \frac{\partial^2 V}{\partial q_1^2}\left(\frac{\mathrm{d}q_1}{\mathrm{d}Q_2}\right)^2 + 2\frac{\partial^2 V}{\partial q_1 \partial Q_2}\left(\frac{\mathrm{d}q_1}{\mathrm{d}Q_2}\right) + \frac{\partial^2 V}{\partial Q_2^2} + \frac{\partial V}{\partial q_1}\frac{\mathrm{d}^2 q_1}{\mathrm{d}Q_2^2}. \tag{17}$$

Evaluating this at F gives the single relevant *stability coefficient*,

$$\frac{\mathrm{d}^2 V}{\mathrm{d}Q_2^2}\bigg|^F = 8EI\frac{\pi^4}{L^3} - \left\{\frac{1}{2}EI\frac{\pi^4}{L^2}\left[\frac{A_0}{L} + \frac{H}{L} + \frac{\pi^2}{8}\left(\frac{A_0}{L}\right)^3 + \frac{3}{8}\pi^2\frac{A_0}{L}\left(\frac{H}{L}\right)^2 + \frac{\pi^2}{2}\left(\frac{H}{L}\right)^3\right] + P\right\}\frac{4}{H},$$

which we set to zero to find the *critical equilibrium state C*. This gives after a little algebra,

$$P^C = \frac{1}{2}EI\frac{\pi^4}{L^2}\left\{3\frac{H}{L} - \frac{A_0}{L} - \frac{\pi^2}{8}\left(\frac{A_0}{L}\right)^3 - \frac{3}{8}\pi^2\frac{A_0}{L}\left(\frac{H}{L}\right)^2 - \frac{\pi^2}{2}\left(\frac{H}{L}\right)^3\right\}.$$

The last three, higher-order, terms can now be safely neglected, since the case of a vanishing leading term ($A_0 = 3H$) is only of passing interest. Thus,

$$P^C = \frac{1}{2}\frac{\pi^4 EI}{L^3}(3H - A_0). \tag{18}$$

We note that, for the range of pre-stress $A_0 \geqslant 3H$ we have $P^C \leqslant 0$, and can tentatively conclude that an arch in this range of heavy pre-stress in the sagging sense cannot be persuaded into a hogging configuration even under zero load. However the transitional case $A_0 = 3H$ is precisely that at which the leading term vanishes, so this conclusion is subject to higher-order effects which we explore no further here.

The arch formed from an initially straight strut. Here $A_0 = 0$, since in the stress-relieved state the arch is straight. Thus we have the simple result for the critical load,

$$P^C = \frac{3}{2}\frac{\pi^4 EIH}{L^3}. \tag{19}$$

The arch with no initial pre-stress. Here the stress-relieved state coincides with the pre-buckled state and $A_0 = -H$. This gives the critical load,

$$P^C = 2\frac{\pi^4 EIH}{L^3}. \tag{20}$$

Higher-order derivatives. It is convenient at this stage to evaluate some of the higher derivatives of V that are required in the post-buckling analysis. This is for two reasons. First they follow naturally from the above differentiation process, but secondly and more important, we thus avoid possible confusion over a significant notational change that follows later. The derivatives are to be evaluated at the critical point C.

Thus, differentiating eqn (17) once with respect to Q_2 and evaluating at C we obtain,

$$\left.\frac{d^3 V}{dQ_2^3}\right|^C = 0.$$

This can be quickly confirmed from symmetry considerations. Differentiating a second time before evaluation gives,

$$\left.\frac{d^4 V}{dQ_2^4}\right|^C = \frac{\partial^4 V}{\partial Q_2^4} + 6\frac{\partial^3 V}{\partial q_1 \partial Q_2^2}\frac{d^2 q_1}{dQ_2^2} + 3\frac{\partial^2 V}{\partial q_1^2}\left(\frac{d^2 q_1}{dQ_2^2}\right)^2 + \left.\frac{\partial V}{\partial q_1}\frac{d^4 q_1}{dQ_2^4}\right|^C, \tag{21}$$

on evaluation at C. Substituting now from the known derivatives we find, perhaps rather surprisingly, that the leading two terms of the right-hand side can be neglected according to the assumptions for moderate rise, $(H/L)^2 \ll 1$, etc. This suggests a result that is later confirmed, that no derivatives of the original potential function (written in terms of q_1 and Q_2) of order higher than quadratic are required to find the post-buckling path curvature. Thus the dominant feature of arch behaviour is the enforced geometry change which arises from the constraint condition.

We thus obtain the result for the fourth derivative

$$\left.\frac{d^4 V}{dQ_2^4}\right|^C = -48\frac{P^C}{H^3} - 24\,EI\frac{\pi^4}{L^3}\frac{A_0}{H^3},$$

and substitution of the general form for P^C of eqn (18) gives,

$$\left.\frac{d^4 V}{dQ_2^4}\right|^C = -72\frac{\pi^4 EI}{H^2 L^3}. \tag{22}$$

We see that the derivative is not dependent on the pre-stress amplitude A_0.

2.5 *Post-buckling analysis*

At this stage it is rewarding to reflect on just what we know about the response of the system. This is summarised schematically in Fig. 3. The constraint condition gives a curved surface in P-Q_1-Q_2 space, on which all allowable equilibrium states must lie. By symmetry, the surface forms part of a complete cylinder which recuts the Q_1 axis at $Q_1 = +H$, but we show here the range of validity of the present analysis in which Q_2 remains single-valued. We note also that the cylinder extends indefinitely in both directions to $P = +\infty$ and $-\infty$.

The fundamental equilibrium state corresponds to a path lying on the cylinder at $Q_1 = -H$ as shown, which becomes unstable at, or just above, the critical point C where $P = P^C$. We

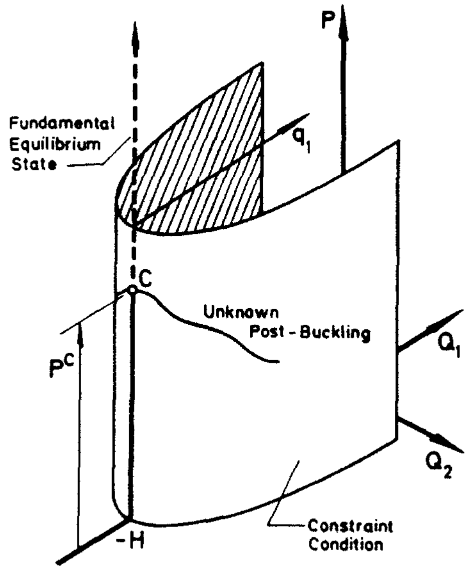

Fig. 3. Information gained from the linear eigenvalue analysis.

cannot yet say anything about the stability of the critical point itself, nor of the nature of the post-buckling equilibrium path which, by a basic theorem of elastic stability must pass through C. These must wait for the post-buckling analysis for clarification.

But the post-buckling path must have a zero slope because of the inherent symmetry, and assuming that it has a non-zero curvature we can therefore expect either a cusp or a dual cusp catastrophe at C. We shall call on the results of our general theory to determine the curvature.

Obligatory notational change. The general results of[5] are derived using a potential function which *here* would be written as,

$$V = V(Q_2, P).$$

This differs in two significant ways from the form used in the eigenvalue analysis. First we have no variation with respect to q_1, so we must assume that at this stage q_1 is fully replaced by its Taylor expansion in terms of Q_2, developed from the constraint condition. Secondly, we see that the load P must now be considered as a variable, since a derivative of V with respect to P is required in the post-buckling analysis. We can, however, make use of the differentiations of the eigenvalue analysis, but *subject to a change of notation.*

Thus, inspecting the earlier differentiation of V, we find that full derivatives with respect to Q_2, those on the l.h.s. side of eqns (16) and (17) for example, must now be written as partial derivatives, since P is no longer assumed constant. On the other hand, partial differentiation with respect to Q_2 by this stage implies varying q_1, since the constraint condition is now automatically satisfied. The meaning of the partial ∂ is thus significantly altered.

Post-buckling path curvature. With the notational change, we have the fourth derivative,

$$\left.\frac{\partial^4 V}{\partial Q_2^4}\right|^C = -72\frac{\pi^4 EI}{H^2 L^3}, \tag{23}$$

from eqn (22), and to find the only outstanding necessary derivative we differentiate eqn (17) with respect to P. This gives,

$$\left.\frac{\partial^3 V}{\partial Q_2^2 \partial P}\right|^C = -\left.\frac{d^2 q_1}{dQ_2^2}\right|^C = -\frac{4}{H}. \tag{24}$$

We can therefore immediately write the curvature of the post-buckling path as

$$\frac{d^2P}{dQ_2^2}\bigg|^C = -\frac{\partial^4 V}{\partial Q_2^4}\bigg/ 3\frac{\partial^3 V}{\partial Q_2^2 \partial P}\bigg|^C = -6\frac{\pi^4 EI}{HL^3}, \tag{25}$$

using ([5] p. 185) with $u_1 = Q_2$. The curvature is negative, so we have an unstable-symmetric point of bifurcation at a dual cusp catastrophe. We note that the curvature is not dependent on the amount of initial pre-stress A_0.

2.6 *Comparison with the true perfect response*

We are now in a position to complete the picture of Fig. 3, and to compare this with the known response of the perfect arch. We consider here only the arch formed from an initially straight strut.

From symmetry, the paths clearly must be as shown on the left of Fig. 4. We have a trivial fundamental solution $Q_1 = -H$, $Q_2 = 0$, together with its mirror image $Q_1 = H$, $Q_2 = 0$, and the constraint condition generates a cylinder extending to $P = \pm\infty$. The unstable post-buckling path forms a closed loop on the cylinder. The system, on reaching the unstable critical state C or its reflection, snaps dynamically to the opposite trivial stable state, in accordance with the constraint condition.

We contrast this picture with the true response of the shallow arch, shown on the right of Fig. 4. The major difference is the form of the fundamental path; now, with the reintroduction of extensibility or higher harmonics, the arch can display limiting behaviour in a symmetric mode ($Q_2 = 0$), giving a highly non-linear, non-trivial path. The limit points are of course already unstable with respect to Q_2. Clearly we cannot now represent the constraint condition in any simple way.

But the post-buckling behaviour is essentially the same, and it is precisely the trivial nature of our fundamental path that gives such simple results. We might suppose that the further apart that the critical bifurcation and limit points are, the more accurate the analysis is likely to be. We note finally that an elastically tied arch can be adjusted, by introducing an extra control parameter, so that the two critical states coincide. This gives the hill-top branching point, which has been investigated recently as an illustration of a hyperbolic umbilic catastrophe [8–10].

2.7 *Imperfection-sensitivity analysis*

The imperfection-sensitivity of the arch is here investigated following Roorda's classic formulation, by off-setting the load a small amount ϵL from the centre-line; here we take ϵ as positive for off-sets in the positive sense of x. The strain energy is clearly unaffected by this

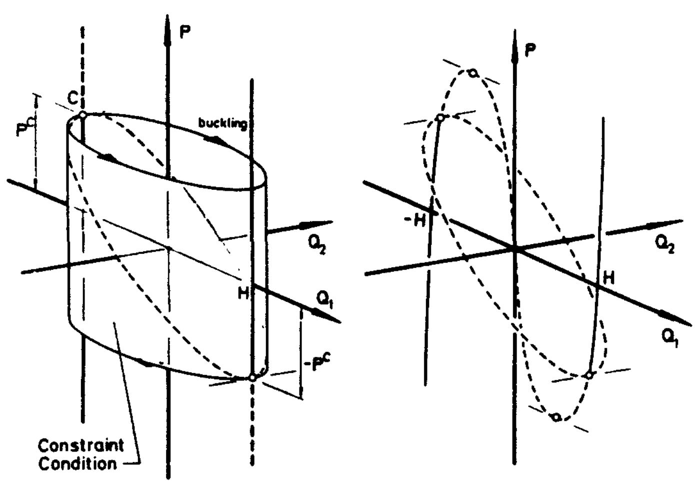

Fig. 4. Comparison between the equilibrium paths of the perfect inextensional model and the true arch response.

introduction of a second control parameter, but the load now drops a further small amount δw which is given by the approximation,

$$\delta w \approx \epsilon L \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{x=L/2}.$$

We can replace this by a rigorous equality, by assuming that the load is applied a distance ϵL along a rigid straight bar, rigidly fixed to the arch at the centre. Alternatively we can assume that ϵ is small enough so that, in comparison with the earlier assumptions, such rigour is unnecessary and the equality is valid for a simple load off-set. We note that experimentally, the former usually applies.

The potential energy of the load is changed by this additional drop, and we have,

$$\begin{aligned} V &= U - P\left[q_1 + \epsilon L w'\left(\frac{L}{2}\right)\right] \\ &= U - P(q_1 - 2\pi\epsilon Q_2), \end{aligned} \tag{26}$$

using the assumed deflected shape (3). Remembering the results from the constraint condition, we can immediately write down the additional necessary coefficient for the imperfection-sensitivity analysis as,

$$\left.\frac{\partial^2 V}{\partial Q_2 \partial \epsilon}\right|^{C} = 2\pi P^C, \tag{27}$$

where C refers to the critical state of the *perfect system* $\epsilon = 0$. Substituting now in the general result of ([5] p. 187) we can obtain the coefficient of the two-thirds power law failure locus,

$$P = P^C - \beta\epsilon^{2/3},$$

where P is now the critical load of an *imperfect system*, as

$$\begin{aligned} \beta &= \left.\left(\frac{\partial^4 V}{\partial Q_2^4}\right)^{1/3}\left(3\frac{\partial^2 V}{\partial Q_2 \partial \epsilon}\right)^{2/3} \Big/ 2\frac{\partial^3 V}{\partial Q_2^2 \partial P}\right|^{C} \\ &= \frac{3}{4} EI \frac{\pi^4}{L^3}\{3\pi^2 H(3H - A_0)^2\}^{1/3}, \end{aligned} \tag{28}$$

on substititution from eqns (23), (24) and (26).

The arch formed from an initially straight strut. Here $A_0 = 0$ as before, and we thus have the two-thirds power-law imperfection-sensitivity relationship,

$$P = P^C - \frac{9}{4}\frac{\pi^4 EIH}{L^3}(\pi\epsilon)^{2/3}.$$

Nondimensionalizing with respect to P^C, given by equation (19), we have,

$$\begin{aligned} \frac{P}{P^C} &= 1 - \frac{3}{2}(\pi\epsilon)^{2/3} \\ &= 1 - 3.22\,\epsilon^{2/3}. \end{aligned} \tag{29}$$

The arch with no initial pre-stress. Here $A_0 = -H$ as before, giving,

$$P = P^C - \frac{3}{2}\frac{\pi^4 EIH}{L^3}(6\pi\epsilon)^{2/3}.$$

Again nondimensionalizing with respect to P^C, given here by eqn (20), we have,

$$\frac{P}{P^C} = 1 - \frac{3}{4}(6\pi^2\epsilon^2)^{1/3}$$
$$= 1 - 2.92\,\epsilon^{2/3}. \tag{30}$$

3. COMPARISON WITH ROORDA'S THEORY

Roorda's theory[1–4], repeated by Huseyin[6], applies only to the pure stress-free arch and produces rather lengthy results, due to the inclusion of axial compressibility. Because of this inclusion, they may seem to be offering a more realistic analysis. This may not necessarily be the case, however, because the pre-buckling nonlinearity of the shallow arches under discussion is primarily due to higher harmonics, rather than the axial compressibility: like us, they only consider two harmonic amplitudes.

If we let EA tend to infinity in Roorda's result for the critical load P^C, we find it then has the same algebraic form as our (20) with the different numerical factor 64π. His result is therefore 1.032 times our own, and the two theories are in close agreement. The very small discrepency is likely to be due to the difference in assumed initial shapes, our arch being sinusoidal and Roorda's circular; we note that the first fourier component of a circular arch would exhibit a slightly different H value from that of the arch itself.

If we treat Roorda's imperfection-sensitivity result in the same way it becomes absolutely identical to our first eqn (30). This is quite remarkable considering the contrasting approaches and the (small) difference in critical loads. Roorda's result has been shown by him to be in good agreement with his test on a stress-free arch.

4. COMPARISON WITH EXPERIMENTS

Our result for P^C as a function of A_0 is shown in Fig. 5 and compared with some available experimental values. Also shown is the result of a simple heuristic 'back of an

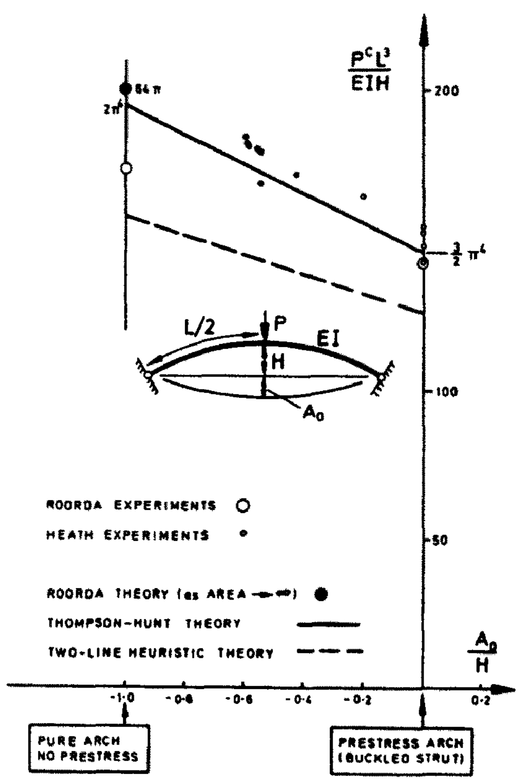

Fig. 5. Comparison between the present theory and the experiments of Roorda[1–4] and N. Heath. The variation of P^C with the magnitude of the prestress.

envelope' approach: this assumes that there is no change of bending moment in the fundamental state, and simply takes moments about the crown for one half of the arch, setting finally the horizontal abutment thrust equal to the second Euler load of the corresponding strut.

The comparison with Roorda's two experiments cannot be expected to be any better than it is, because our calculations are based only on his published *nominal* cross-section dimensions (the I of the cross-section is of course very sensitive to the assumed depth) and on a nominal Young's Modulus for steel $E = 30 \times 10^6$ lb/in^2. Values used were for the pure un-stressed arch, H = 1.55 in, L = 24 in, with a rectangular cross-section of depth 1/32 in and breadth 1 in: for the buckled strut, H = 1.5 in, L = 24 in, with a rectangular cross-section of depth 1/16 in and breadth 1 in. We have ignored any $(1-\nu^2)$ Poisson's ratio effect in both cases.

The experiments of N. Heath on a series of shallow prestressed arches were conducted as a preliminary investigation at Imperial College London under the supervision of the second author: better corelation could be expected here, because the EI of the section was obtained experimentally by a direct bending test. The trend of the experimental results nicely confirms the validity of our theory.

Our imperfection-sensitivity results were also found to be in good agreement with the tests of N. Heath, and we show finally in Fig. 6 a comparison with one of the original experiments of Roorda[1–4]. This was on an arch made from a buckled strut, and has not previously been compared with a theoretical solution. Agreement is quite good on both diagrams, and on the lower one is certainly better than that obtained if we just ignore the prestress and use the results of a pure arch. In the top load deflection diagram the validity of our theory is restricted to the fairly small deflection range, as we would expect from a local bifurcational analysis. The small discrepancy in the lower imperfection-sensitivity diagram could easily be due to the difficulty in assessing the precise top of the cusp.

5. CONCLUDING REMARKS

The simple inextensional theory that we have presented here for the buckling and post-buckling of shallow prestressed arches seems to agree well with the existing theoretical and experimental evidence. Useful neat equations for the bifurcation load and the two-thirds power-law imperfection-sensitivity are given. It is suggested that more experiments are needed, with careful calibration tests, to explore more fully the complete range of prestress magnitude.

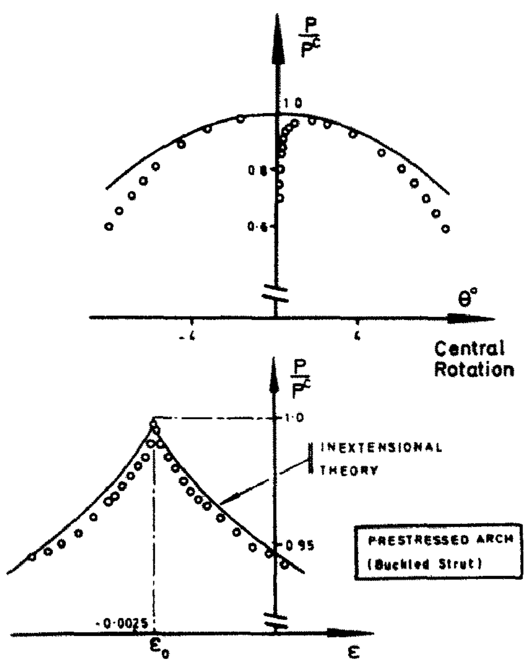

Fig. 6. Comparison between the present theory and the experiment of Roorda[1–4] on a prestressed arch made from a buckled strut.

Axial compressibility forms an integral part of most studies of shallow arches[11–14]. We note however that an inextensional form is taken to define the perfect system, and compressibility included as an extra degree of control, in a recent contribution of Oran[15], but for a pressure-loaded arch with clamped ends.

REFERENCES

1. J. Roorda, *The instability of imperfect elastic structures.* Ph.D. Thesis, University College, London, 1965.
2. J. Roorda, Stability of structures with small imperfections. *J. Engng Mech. Div. Am. Soc. Civ. Engrs* 91, 87 (1965).
3. J. Roorda, An experience in equilibrium and stability, Tech. Note No. 3, Solid Mechanics Division, University of Waterloo, Ontario, Canada, May 1971.
4. J. Roorda, *Buckling of Elastic Structures.* Special Publication Series, Solid Mechanics Division, University of Waterloo Press, Waterloo (1980).
5. J. M. T. Thompson and G. W. Hunt, *A General Theory of Elastic Stability.* Wiley, London (1973).
6. K. Huseyin, *Non-linear Theory of Elastic Stability.* Noordhoff, Leyden (1974).
7. E. C. Zeeman, *Catastrophe Theory: Selected Papers 1972–1977.* Addison-Wesley, London, (1977).
8. J. M. T. Thompson and G. W. Hunt, The instability of evolving systems. *Interdisciplinary Sci. Rev.* 2, 240 (1977).
9. J. M. T. Thompson, *Instabilities and Catastrophes in Science and Engineering.* Wiley, Chichester (1982).
10. J. M. T. Thompson and G. W. Hunt, *Elastic Instability Phenomena.* Wiley, Chichester, to appear.
11. H. L. Schreyer and E. F. Masur, Buckling of shallow arches. *J. Engng Mech. Div. Am. Soc. Civ. Engrs* 92, 1 (1966).
12. E. F. Masur and D. L. C. Lo, The shallow arch—general buckling, postbuckling, and imperfection analysis. *J. Struct. Mech.* 1, 1 (1972).
13. C. L. Dym, Bifurcation analysis for shallow arches. *J. Engng Mech. Div. Am. Soc. Civ. Engrs* 99, 287 (1973).
14. C. L. Dym, *Stability Theory and its Applications to Structural Mechanics.* Noordhoff, Leyden (1974).
15. C. Oran, General imperfection analysis of shallow arches. *J. Engng Mech. Div. Am. Soc. Civ. Engrs* 106, 1175 (1980).